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Reflection coefficients for weak anisotropic media

Matthias Zillmer,[1] Dirk Gajewski[1] and Boris M. Kashtan[2]
[1] *Institute of Geophysics, University of Hamburg, Bundesstrasse* 55, 20146 *Hamburg, Germany*
[2] *Institute for Physics, St. Petersburg State University, Ulyanovskaya* 1, *Petrodvorets, St. Petersburg* 198904, *Russia*

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SUMMARY

The interaction of plane elastic waves with a plane boundary between two anisotropic elastic half-spaces is investigated. The anisotropy dealt with in this study is of a general type. Explicit expressions for energy-related reflection and transmission coefficients are derived. They represent an approximation which is valid for a small deviation of the elastic parameters from isotropy.

Classical perturbation theory is applied on a 6×6 non-symmetric real eigenvalue problem to calculate first-order corrections for the polarization and stress of the plane waves. The explicit solution of the isotropic problem is used as a reference case. Degenerate perturbation theory is used to consider the splitting of the isotropic *S*-wave into two anisotropic *qS*-waves. The boundary conditions for two half-spaces in welded contact lead to a 6×6 system of linear equations. A correction to the isotropic solution is calculated by linearization. The resultant coefficients are functions of horizontal slowness, Lamé parameters and densities of the reference media, and of the perturbation of the elasticity tensors from isotropy.

Key words: anisotropy, perturbation methods.

INTRODUCTION

The calculation of reflection coefficients for the interaction of plane waves with an interface between two general anisotropic half-spaces requires the numerical solution of a 6×6 eigenvalue problem and the subsequent solution of a 6×6 system of linear equations (Fedorov 1968; Keith & Crampin 1977; Rokhlin, Bolland & Adler 1986). In cases of isotropic or transversely isotropic media, both parts of the problem are simplified because of the decoupling of the qP–qSV from the SH plane-wave motion. This makes it possible to obtain an explicit solution by solving two subset problems. Exact analytic expressions for reflection coefficients can be calculated for such a problem (Daley & Hron 1977).

More general types of anisotropy are necessary to describe media which contain cracks with an arbitrary orientation of the crack system, or media which describe a system of layers not horizontally oriented. For these types of media the reflection problem has to be solved numerically. One can either use the Christoffel equation or the system matrix of an ordinary differential equation system to calculate the polarization and stress of the plane waves for a fixed horizontal slowness. The problem is discussed in detail in Fryer & Frazer (1987).

In another paper, the same authors discuss how to use perturbation theory to get the stress–displacement vectors of the waves based on an anisotropic reference medium (Frazer & Fryer 1989).

Here, we use the isotropic eigenvalue problem with its explicit solution as a reference case to calculate the stress–displacement vectors of the plane waves for general anisotropy. This is accomplished with the aid of classical perturbation theory (Wilkinson 1965). The existence of an orthogonality relation is important for this technique. It is known that this relation exists for general anisotropy (Ingebrigtsen & Tonning 1969; Garmany 1983). The linear system which describes the boundary conditions is solved for a perturbed coefficient matrix and perturbed right-hand-side vector by linearization.

STATEMENT OF THE PROBLEM

The eigenvalue problem

We work with a Cartesian coordinate system with the z-axis vertical to the interface pointing upwards and the x-axis parallel to the projection of the slowness of the incident wave on the interface. With this special coordinate system we set $\partial/\partial y = 0$.

The equations of motion and the constitutive equations for linear elasticity are

$$\frac{\partial \hat{\boldsymbol{\sigma}}_3}{\partial z} = \rho \frac{\partial^2 \hat{\mathbf{u}}}{\partial t^2} - \frac{\partial \hat{\boldsymbol{\sigma}}_1}{\partial x}, \tag{1}$$

$$\hat{\boldsymbol{\sigma}}_1 = \mathbf{C}_{11} \frac{\partial \hat{\mathbf{u}}}{\partial x} + \mathbf{C}_{13} \frac{\partial \hat{\mathbf{u}}}{\partial z}, \tag{2}$$

$$\hat{\boldsymbol{\sigma}}_3 = \mathbf{C}_{31} \frac{\partial \hat{\mathbf{u}}}{\partial x} + \mathbf{C}_{33} \frac{\partial \hat{\mathbf{u}}}{\partial z},$$

with elasticity tensor $(\mathbf{C}_{ik})_{jl} = c_{ijkl}$, stress $\hat{\boldsymbol{\sigma}}_j = \hat{\sigma}_{ij}$, displacement $\hat{\mathbf{u}}$ and density ρ.

These equations can be written in matrix form:

$$\frac{\partial}{\partial z}\begin{pmatrix}\hat{\mathbf{u}}\\ \hat{\boldsymbol{\sigma}}_3\end{pmatrix}=\begin{pmatrix}\hat{\mathbf{A}}_{11} & \hat{\mathbf{A}}_{12}\\ \hat{\mathbf{A}}_{21} & \hat{\mathbf{A}}_{22}\end{pmatrix}\begin{pmatrix}\hat{\mathbf{u}}\\ \boldsymbol{\sigma}_3\end{pmatrix}, \tag{3}$$

with

$$\hat{\mathbf{A}}_{11}=-\mathbf{C}_{33}^{-1}\mathbf{C}_{31}\frac{\partial}{\partial x},$$

$$\hat{\mathbf{A}}_{12}=\mathbf{C}_{33}^{-1},$$

$$\hat{\mathbf{A}}_{21}=\rho\mathbf{I}\frac{\partial^2}{\partial t^2}-\frac{\partial}{\partial x}\left[(\mathbf{C}_{11}-\mathbf{C}_{13}\mathbf{C}_{33}^{-1}\mathbf{C}_{31})\frac{\partial}{\partial x}\right.,$$

$$\hat{\mathbf{A}}_{22}=-\frac{\partial}{\partial x}\left[\mathbf{C}_{13}\mathbf{C}_{33}^{-1}\right..$$

We apply the following Fourier transformation with an integration over frequency ω and horizontal slowness p:

$$\begin{pmatrix}\hat{\mathbf{u}}\\ \hat{\boldsymbol{\sigma}}_3\end{pmatrix}(t,x,z)=\mathscr{Re}\left\{\frac{1}{\pi}\int_0^{\infty}d\omega\int_{-\infty}^{\infty}dp\begin{pmatrix}\mathbf{u}\\ \boldsymbol{\sigma}_3\end{pmatrix}(\omega,p,z)\times\exp[-i\omega(t-px)]\right\}. \tag{4}$$

Then, in the transform domain eq. (3) reads

$$\frac{d}{dz}\begin{pmatrix}\mathbf{u}\\ \boldsymbol{\tau}\end{pmatrix}=i\omega\begin{pmatrix}\mathbf{A}_{11} & \mathbf{A}_{12}\\ \mathbf{A}_{21} & \mathbf{A}_{22}\end{pmatrix}\begin{pmatrix}\mathbf{u}\\ \boldsymbol{\tau}\end{pmatrix}, \tag{5}$$

with

$$\mathbf{A}_{11}=-p\mathbf{C}_{33}^{-1}\mathbf{C}_{31},$$

$$\mathbf{A}_{12}=-\mathbf{C}_{33}^{-1},$$

$$\mathbf{A}_{21}=p^2(\mathbf{C}_{11}-\mathbf{C}_{13}\mathbf{C}_{33}^{-1}\mathbf{C}_{31})-\rho\mathbf{I},$$

$$\mathbf{A}_{22}=-p\mathbf{C}_{13}\mathbf{C}_{33}^{-1},$$

and a scaled normal-stress vector $\boldsymbol{\tau}=\boldsymbol{\sigma}_3/(-i\omega)$. We call the system matrix in eq. (5) $\mathbf{A}$.

The solution of (5) is given by

$$\begin{pmatrix}\mathbf{u}\\ \boldsymbol{\tau}\end{pmatrix}(\omega,p,z)=\sum_{l=1}^{6}U_l\mathbf{x}_l(p)\exp[i\omega q_l(p)z], \tag{6}$$

with $\mathbf{x}_l$ the right-hand eigenvectors of the matrix $\mathbf{A}$ and q_l the corresponding eigenvalues.

The eigenvalue problem for $\mathbf{A}$ is written

$$\mathbf{X}^{-1}\mathbf{A}\mathbf{X}=\mathbf{Q}. \tag{7}$$

The eigenvalues represent the vertical slownesses of the upgoing and downgoing qP, $qS1$ and $qS2$ waves. The right eigenvectors $\mathbf{X}$ represent the stress–displacement vectors of the waves. The left eigenvectors coincide with the right ones after a proper normalization and after reordering the stress and displacement part. This is due to the following property of the matrix $\mathbf{A}$:

$$\mathbf{KA}=(\mathbf{KA})^{\mathrm{T}},\qquad \mathbf{K}=\begin{pmatrix}\mathbf{0} & \mathbf{I}\\ \mathbf{I} & \mathbf{0}\end{pmatrix}. \tag{8}$$

We define the following inner product for the right eigenvectors:

$$(\mathbf{x}_m,\mathbf{x}_n)=\mathbf{x}_m^{\mathrm{T}}\mathbf{K}\mathbf{x}_n=\mathbf{u}_m\boldsymbol{\tau}_n+\boldsymbol{\tau}_m\mathbf{u}_n,\qquad m,n=1\ldots6 \tag{9}$$

and introduce the norm

$$\|\mathbf{x}_m\|=\sqrt{(\mathbf{x}_m,\mathbf{x}_m)}. \tag{10}$$

With this normalization the right eigenvectors are orthonormal (Ingebrigtsen & Tonning 1969):

$$(\mathbf{x}_m,\mathbf{x}_n)=\delta_{mn},\qquad m,n=1\ldots6 \tag{11}$$

and the inverse matrix can be calculated from

$$\mathbf{X}^{-1}=\mathbf{X}^{\mathrm{T}}\mathbf{K}. \tag{12}$$

For real eigenvalues q_l this normalization is related to the vertical energy flux (Garmany 1983).

The solution of (7) can be calculated explicitly for the case of an isotropic medium. Because we use it as a reference solution we denote it with the upper index (0):

$$\mathbf{X}^{(0)-1}\mathbf{A}^{(0)}\mathbf{X}^{(0)}=\mathbf{Q}^{(0)}. \tag{13}$$

The form of $\mathbf{A}^{(0)}$ and the results for $\mathbf{X}^{(0)}$, $\mathbf{X}^{(0)-1}$ and $\mathbf{Q}^{(0)}$ are given in the appendix.

We now use classical perturbation theory (Wilkinson 1965) for the eigenvalue problem (7) of the matrix $\mathbf{A}$. We consider an elasticity tensor which consists of an isotropic part, $c_{ijkl}^{(0)}=(\lambda/\rho)\delta_{ij}\delta_{kl}+(\mu/\rho)(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})$, and a small anisotropic perturbation, $\epsilon c_{ijkl}^{(1)}$:

$$c_{ijkl}=c_{ijkl}^{(0)}+\epsilon c_{ijkl}^{(1)}. \tag{14}$$

ϵ is a small perturbation parameter.

We can separate the matrix $\mathbf{A}$ into

$$\mathbf{A}=\mathbf{A}^{(0)}+\epsilon\mathbf{A}^{(1)}, \tag{15}$$

where the matrix $\mathbf{A}^{(1)}$ is given by

$$\mathbf{A}^{(1)}=\begin{pmatrix}\mathbf{A}_{11}^{(1)} & \mathbf{A}_{12}^{(1)}\\ \mathbf{A}_{21}^{(1)} & \mathbf{A}_{22}^{(1)}\end{pmatrix} \tag{16}$$

with

$$\mathbf{A}_{11}^{(1)}=-p(\mathbf{C}_{33}^{(0)-1}\mathbf{C}_{31}^{(1)}-\mathbf{C}_{33}^{(0)-1}\mathbf{C}_{33}^{(1)}\mathbf{C}_{33}^{(0)-1}\mathbf{C}_{31}^{(0)}),$$

$$\mathbf{A}_{12}^{(1)}=-\mathbf{C}_{33}^{(0)-1}\mathbf{C}_{33}^{(1)}\mathbf{C}_{33}^{(0)-1},$$

$$\mathbf{A}_{21}^{(1)}=p^2(\mathbf{C}_{11}^{(1)}-\mathbf{C}_{13}^{(1)}\mathbf{C}_{33}^{(0)-1}\mathbf{C}_{31}^{(0)}-\mathbf{C}_{13}^{(0)}\mathbf{C}_{33}^{(0)-1}\mathbf{C}_{31}^{(1)}+\mathbf{C}_{13}^{(0)}\mathbf{C}_{33}^{(0)-1}\mathbf{C}_{33}^{(1)}\mathbf{C}_{33}^{(0)-1}\mathbf{C}_{31}^{(0)}), \tag{17}$$

$$\mathbf{A}_{22}^{(1)}=-p(\mathbf{C}_{13}^{(1)}\mathbf{C}_{33}^{(0)-1}-\mathbf{C}_{13}^{(0)}\mathbf{C}_{33}^{(0)-1}\mathbf{C}_{33}^{(1)}\mathbf{C}_{33}^{(0)-1}).$$

The elements of the perturbation part $\mathbf{A}^{(1)}$ are given explicitly in the appendix.

We calculate the elements of the perturbation matrix $\mathbf{S}$:

$$\mathbf{S}=\mathbf{X}^{(0)-1}\mathbf{A}^{(1)}\mathbf{X}^{(0)}=\mathbf{X}^{(0)\mathrm{T}}\mathbf{K}\mathbf{A}^{(1)}\mathbf{X}^{(0)}. \tag{18}$$

$\mathbf{S}$ is symmetric because of the symmetry of $\mathbf{KA}^{(1)}$. The elements of the perturbation matrix are given in the appendix.

In our isotropic reference case there are two pairs of eigenvectors with the same eigenvalue: $q_2^{(0)}=q_3^{(0)}=q_S$ and $q_5^{(0)}=q_6^{(0)}=-q_S$. Due to the fact that only one upgoing and one downgoing S-wave exist we have to use degenerate perturbation theory. These double eigenvalues separate in the anisotropic case to give the upgoing and downgoing $qS1$ and $qS2$ waves.

We want to calculate corrections of the first order to the eigenvalues $q_l^{(0)}$ and eigenvectors $\mathbf{x}_l^{(0)}$; because of the degeneracy we have to consider terms up to the second order. We use the expansion

$$(\mathbf{A}^{(0)} + \epsilon\mathbf{A}^{(1)}) \sum_{k=1}^{6} (f_{mk}^{(0)} + \epsilon f_{mk}^{(1)} + \epsilon^2 f_{mk}^{(2)})\mathbf{x}_k^{(0)}$$
$$= (q_m^{(0)} + \epsilon q_m^{(1)} + \epsilon^2 q_m^{(2)})$$
$$\times \sum_{k=1}^{6} (f_{mk}^{(0)} + \epsilon f_{mk}^{(1)} + \epsilon^2 f_{mk}^{(2)})\mathbf{x}_k^{(0)}, \qquad m = 1\ldots6. \tag{19}$$

We compare terms of equal order in ϵ on both sides of the equation and use the orthogonality of the basis $\mathbf{x}_l^{(0)}$.

From a comparison of zero-order terms we have

$$f_{mn}^{(0)} = \delta_{mn}, \tag{20}$$
$$m, n = 1\ldots6, \qquad \{mn\}\not\in\{22, 23, 32, 33, 55, 56, 65, 66\}.$$

From a comparison of the first-order terms we obtain the following complex symmetric 2×2 eigenvalue problem:

$$\begin{pmatrix} S_{22} - q_m^{(1)} & S_{23} \\ S_{23} & S_{33} - q_m^{(1)} \end{pmatrix} \begin{pmatrix} f_{m2}^{(0)} \\ f_{m3}^{(0)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad m = 2, 3. \tag{21}$$

The solution is

$$q_{2,3}^{(1)} = \frac{1}{2}[S_{22} + S_{33} \pm \sqrt{(S_{22} - S_{33})^2 + 4S_{23}^2}],$$
$$f_{22}^{(0)} = -\frac{S_{33} - q_2^{(1)}}{\sqrt{(S_{33} - q_2^{(1)})^2 + S_{23}^2}},$$
$$f_{23}^{(0)} = \frac{S_{23}}{\sqrt{(S_{33} - q_2^{(1)})^2 + S_{23}^2}}, \tag{22}$$
$$f_{33}^{(0)} = f_{22}^{(0)}, \qquad f_{32}^{(0)} = -f_{23}^{(0)}.$$

For the square roots in eq. (22) we choose $\mathcal{I}m \geq 0$. We have the same problem and solution for $m = 5, 6$ instead of $m = 2, 3$. We note that, for real q_S, eq. (21) is a real eigenvalue problem.

We construct a new basis of the isotropic reference case (Landau & Lifschitz 1958):

$$\mathbf{Z}^{(0)} = \mathbf{X}^{(0)}\mathbf{F}. \tag{23}$$

with

$$\mathbf{F} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & f_{22}^{(0)} & f_{32}^{(0)} & 0 & 0 & 0 \\ 0 & f_{23}^{(0)} & f_{33}^{(0)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & f_{55}^{(0)} & f_{65}^{(0)} \\ 0 & 0 & 0 & 0 & f_{56}^{(0)} & f_{66}^{(0)} \end{pmatrix}, \tag{24}$$

$$\mathbf{F}^{\mathrm{T}} = \mathbf{F}^{-1}. \tag{25}$$

Eq. (23) is a rotation of the SH and SV vectors of the old basis. After rotation we have a zero-order approximation of the anisotropic $qS1$ and $qS2$ stress–displacement vectors. With the new basis $\mathbf{z}_l^{(0)}$ we recalculate a perturbation matrix $\mathbf{V}$:

$$\mathbf{V} = \mathbf{F}^{\mathrm{T}}\mathbf{S}\mathbf{F}. \tag{26}$$

$\mathbf{V}$ is symmetric, $V_{23} = 0$ and $V_{56} = 0$.

We repeat the approach of eq. (19):

$$(\mathbf{A}^{(0)} + \epsilon\mathbf{A}^{(1)}) \sum_{k=1}^{6} (h_{mk}^{(0)} + \epsilon h_{mk}^{(1)} + \epsilon^2 h_{mk}^{(2)})\mathbf{z}_k^{(0)}$$
$$= (q_m^{(0)} + \epsilon q_m^{(1)} + \epsilon^2 q_m^{(2)})$$
$$\times \sum_{k=1}^{6} (h_{mk}^{(0)} + \epsilon h_{mk}^{(1)} + \epsilon^2 h_{mk}^{(2)})\mathbf{z}_k^{(0)}, \qquad m = 1\ldots6. \tag{27}$$

The first-order correction to the eigenvalues is given by the diagonal elements of the matrix $\mathbf{V}$:

$$q_m^{(1)} = V_{mm}, \qquad m = 1\ldots6. \tag{28}$$

We can use this result to calculate the slowness surface of the anisotropic medium.

From a comparison of terms up to the second order in ϵ we obtain the correction for the eigenvectors $\mathbf{z}_l^{(0)}$:

$$h_{mn}^{(0)} = \delta_{mn}, \qquad m, n = 1\ldots6,$$
$$h_{mm}^{(1)} = 0, \qquad m = 1\ldots6,$$
$$h_{mn}^{(1)} = \frac{V_{nm}}{q_m^{(0)} - q_n^{(0)}}, \qquad m, n = 1\ldots6, \qquad m \neq n,$$
$$\{mn\}\not\in\{23, 32, 56, 65\},$$
$$h_{23}^{(1)} = \frac{V_{31}h_{21}^{(1)} + V_{34}h_{24}^{(1)} + V_{35}h_{25}^{(1)} + V_{36}h_{26}^{(1)}}{V_{22} - V_{33}},$$
$$h_{32}^{(1)} = \frac{V_{21}h_{31}^{(1)} + V_{24}h_{34}^{(1)} + V_{25}h_{35}^{(1)} + V_{26}h_{36}^{(1)}}{V_{33} - V_{22}}, \tag{29}$$
$$h_{56}^{(1)} = \frac{V_{61}h_{51}^{(1)} + V_{62}h_{52}^{(1)} + V_{63}h_{53}^{(1)} + V_{64}h_{54}^{(1)}}{V_{55} - V_{66}},$$
$$h_{65}^{(1)} = \frac{V_{51}h_{61}^{(1)} + V_{52}h_{62}^{(1)} + V_{53}h_{63}^{(1)} + V_{54}h_{64}^{(1)}}{V_{66} - V_{55}}.$$

The stress–displacement vectors of the anisotropic medium in first-order perturbation theory are finally given by

$$\mathbf{Z} = \mathbf{Z}^{(0)} + \epsilon\mathbf{Z}^{(1)} = \mathbf{X}^{(0)}\mathbf{F}(\mathbf{I} + \epsilon\mathbf{H}), \tag{30}$$

with

$$\mathbf{H} = \begin{pmatrix} 0 & h_{21}^{(1)} & h_{31}^{(1)} & h_{41}^{(1)} & h_{51}^{(1)} & h_{61}^{(1)} \\ h_{12}^{(1)} & 0 & h_{32}^{(1)} & h_{42}^{(1)} & h_{52}^{(1)} & h_{62}^{(1)} \\ h_{13}^{(1)} & h_{23}^{(1)} & 0 & h_{43}^{(1)} & h_{53}^{(1)} & h_{63}^{(1)} \\ h_{14}^{(1)} & h_{24}^{(1)} & h_{34}^{(1)} & 0 & h_{54}^{(1)} & h_{64}^{(1)} \\ h_{15}^{(1)} & h_{25}^{(1)} & h_{35}^{(1)} & h_{45}^{(1)} & 0 & h_{65}^{(1)} \\ h_{16}^{(1)} & h_{26}^{(1)} & h_{36}^{(1)} & h_{46}^{(1)} & h_{56}^{(1)} & 0 \end{pmatrix} = -\mathbf{H}^{\mathrm{T}}. \tag{31}$$

The approach described is valid as long as the factors $h_{mn}^{(1)}$ are small. We know from numerical experiments that the result is acceptable for $|h_{mn}^{(1)}| < 0.15$. This limit is exceeded for values of horizontal slowness p near the critical angle of the isotropic reference case ($q_P \approx 0$, $q_S \approx 0$) and near shear-wave singularities. In both cases the denominators in the expressions (29) for $h_{mn}^{(1)}$ go to zero.

The linear system

The boundary conditions for two solid half-spaces in welded contact require the continuity of displacement and normal stress. This results in a system of six linear equations whose coefficient matrix $\mathbf{N}$ is given by three stress–displacement vectors of each half-space.

$$\mathbf{N}\mathbf{r} = \mathbf{n}\,, \tag{32}$$

$$\mathbf{N} = [\,-\mathbf{z}_1^{[1]},\ -\mathbf{z}_2^{[1]},\ -\mathbf{z}_3^{[1]}, \mathbf{z}_4^{[2]}, \mathbf{z}_5^{[2]}, \mathbf{z}_6^{[2]}\,]\,. \tag{33}$$

We denote the upper half-space, which contains the incident wave, by the index **[1]** and the lower half-space by **[2]**. The right-hand side of the system is given by the stress–displacement vector $\mathbf{n}$ of the incident wave:

$$\mathbf{n} = \mathbf{z}_m^{[1]}\,, \qquad m = 4, 5, 6\,. \tag{34}$$

The chosen coordinate system coincides with the plane of incidence and the problem for the isotropic reference case decouples into a 4×4 and a 2×2 system which can be solved explicitly. We use the following notation for incident *P*, *SH* and *SV* waves respectively:

$$\begin{aligned}
&\mathbf{N}^{(0)}\mathbf{r}_P = \mathbf{x}_4^{(0)[1]}\,,\\
&\mathbf{N}^{(0)}\mathbf{r}_{SH} = \mathbf{x}_5^{(0)[1]}\,,\\
&\mathbf{N}^{(0)}\mathbf{r}_{SV} = \mathbf{x}_6^{(0)[1]}\,,\\
&\mathbf{r}_P = (R_{PP}, R_{PSH}, R_{PSV}, T_{PP}, T_{PSH}, T_{PSV})^{\mathrm{T}}\,,\\
&\mathbf{r}_{SH} = (R_{SHP}, R_{SHSH}, R_{SHSV}, T_{SHP}, T_{SHSH}, T_{SHSV})^{\mathrm{T}}\,,\\
&\mathbf{r}_{SV} = (R_{SVP}, R_{SVSH}, R_{SVSV}, T_{SVP}, T_{SVSH}, T_{SVSV})^{\mathrm{T}}\,.
\end{aligned} \tag{35}$$

The inverse matrix $\mathbf{N}^{(0)-1}$ and the isotropic reflection coefficients are given in the appendix.

When we rotate the isotropic **S**-vectors as described in the last section we obtain a zero-order approximation for the anisotropic reflection coefficients.

We use the matrix **G**:

$$\mathbf{G} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0\\
0 & f_{22}^{(0)[1]} & -f_{23}^{(0)[1]} & 0 & 0 & 0\\
0 & f_{23}^{(0)[1]} & f_{22}^{(0)[1]} & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & f_{55}^{(0)[2]} & -f_{56}^{(0)[2]}\\
0 & 0 & 0 & 0 & f_{56}^{(0)[2]} & f_{55}^{(0)[2]}
\end{pmatrix}, \tag{36}$$

$$\mathbf{G}^{-1} = \mathbf{G}^{\mathrm{T}}\,. \tag{37}$$

An incident *qP* wave is given to zero order by an incident *P* wave, and incident *qS*1 and *qS*2 waves are composed from *SH* and *SV* waves:

$$\begin{aligned}
&\mathbf{N}^{(0)}\mathbf{G}\mathbf{G}^{\mathrm{T}}\mathbf{r}_P = \mathbf{x}_4^{(0)[1]}\,,\\
&\mathbf{N}^{(0)}\mathbf{G}\mathbf{G}^{\mathrm{T}}(f_{55}^{(0)[1]}\mathbf{r}_{SH} + f_{56}^{(0)[1]}\mathbf{r}_{SV}) =_{55}^{(0)[1]}\mathbf{x}_5^{(0)[1]} + f_{56}^{(0)[1]}\mathbf{x}_6^{(0)[1]}\,,\\
&\mathbf{N}^{(0)}\mathbf{G}\mathbf{G}^{\mathrm{T}}(f_{65}^{(0)[1]}\mathbf{r}_{SH} + f_{66}^{(0)[1]}\mathbf{r}_{SV}) = f_{65}^{(0)[1]}\mathbf{x}_5^{(0)[1]} + f_{66}^{(0)[1]}\mathbf{x}_6^{(0)[1]}\,.
\end{aligned} \tag{38}$$

Thus we obtain a zero-order approximation of the anisotropic coefficients:

$$\begin{aligned}
&\mathbf{r}_{qP}^{(0)} = \mathbf{G}^{\mathrm{T}}\mathbf{r}_P\,,\\
&\mathbf{r}_{qS1}^{(0)} = \mathbf{G}^{\mathrm{T}}(f_{55}^{(0)[1]}\mathbf{r}_{SH} + f_{56}^{(0)[1]}\mathbf{r}_{SV})\,,\\
&\mathbf{r}_{qS2}^{(0)} = \mathbf{G}^{\mathrm{T}}(f_{65}^{(0)[1]}\mathbf{r}_{SH} + f_{66}^{(0)[1]}\mathbf{r}_{SV})\,.
\end{aligned} \tag{39}$$

The *P*-wave reflection and transmission coefficients are identical to the isotropic ones to zero order.

To calculate reflection coefficients to first order, $\mathbf{r}^{(0)} + \epsilon\mathbf{r}^{(1)}$, we use

$$\mathbf{N}^{[1]} = [\,-\mathbf{z}_1^{(1)[1]},\ -\mathbf{z}_2^{(1)[1]},\ -\mathbf{z}_3^{(1)[1]}, \mathbf{z}_4^{(1)[2]}, \mathbf{z}_5^{(1)[2]}, \mathbf{z}_6^{(1)[2]}\,]\,, \tag{40}$$

and

$$\mathbf{n}^{(1)} = \mathbf{z}_m^{(1)[1]}\,, \quad m = 4, 5, 6\,. \tag{41}$$

We delete the term ϵ^2 in the equation

$$(\mathbf{N}^{(0)}\mathbf{G} + \epsilon\mathbf{N}^{(1)})(\mathbf{r}^{(0)} + \epsilon\mathbf{r}^{(1)}) = \mathbf{n}^{(0)} + \epsilon\mathbf{n}^{(1)}\,. \tag{42}$$

We obtain the first-order correction to the zero-order result of eq. (39):

$$\begin{aligned}
&\mathbf{r}_{qP}^{(1)} = \mathbf{G}^{\mathrm{T}}\mathbf{N}^{(0)-1}(\mathbf{z}_4^{(1)[1]} - \mathbf{N}^{(1)}\mathbf{r}_{qP}^{(0)})\,,\\
&\mathbf{r}_{qS1}^{(1)} = \mathbf{G}^{\mathrm{T}}\mathbf{N}^{(0)-1}(\mathbf{z}_5^{(1)[1]} - \mathbf{N}^{(1)}\mathbf{r}_{qS1}^{(0)})\,,\\
&\mathbf{r}_{qS2}^{(1)} = \mathbf{G}^{\mathrm{T}}\mathbf{N}^{(0)-1}(\mathbf{z}_6^{(1)[1]} - \mathbf{N}^{(1)}\mathbf{r}_{qS2}^{(0)})\,.
\end{aligned} \tag{43}$$

EXAMPLES

To demonstrate the quality of the approximate result (39) and (43) we recalculate an example which was used by Keith & Crampin (1977) to describe the reflection between the Earth's crust and mantle. They published some systematic plots of reflection coefficients for an isotropic–anisotropic interface. In their Fig. 8(a) they show the coefficients for a *P*-wave incident in an isotropic half-space on a boundary to a hexagonal half-space with a horizontal symmetry axis. We repeat the calculation for the profile with an azimuth of 45° out of the symmetry plane and compare the exact result with our approximation.

The Lamé parameters of the isotropic upper half-space are $\lambda_1 = 100.67$ GPa and $\mu_1 = 78.34$ GPa; the density is $\rho_1 = 3.4$ g cm^{-3}. In our specific coordinate system the elastic tensor of the lower half-space $c_{ijkl}^{[2]}$ is in units of GPa:

$$\begin{pmatrix}
228.29 & 82.49 & 76.50 & 0.00 & 0.00 & 15.00\\
82.49 & 228.29 & 76.50 & 0.00 & 0.00 & 15.00\\
76.50 & 76.50 & 200.77 & 0.00 & 0.00 & 3.51\\
0.00 & 0.00 & 0.00 & 68.40 & 4.51 & 0.00\\
0.00 & 0.00 & 0.00 & 4.5168.40 & 0.00 & \\
15.00 & 15.00 & 3.51 & 0.00 & 0.00 & 75.39
\end{pmatrix},$$

and the density is $\rho_2 = 3.3$ g cm^{-3}. To construct an isotropic reference medium we use the Voigt average, which is $\lambda_2 = 78.33$ GPa and $\mu_2 = 70.56$ GPa. The perturbation is then given by $\epsilon c_{ijkl}^{(1)} = c_{ijkl} - c_{ijkl}^{(0)}$.

The solid lines in Fig. (1) represent the exact numerical coefficients, and the dotted lines represent the approximation

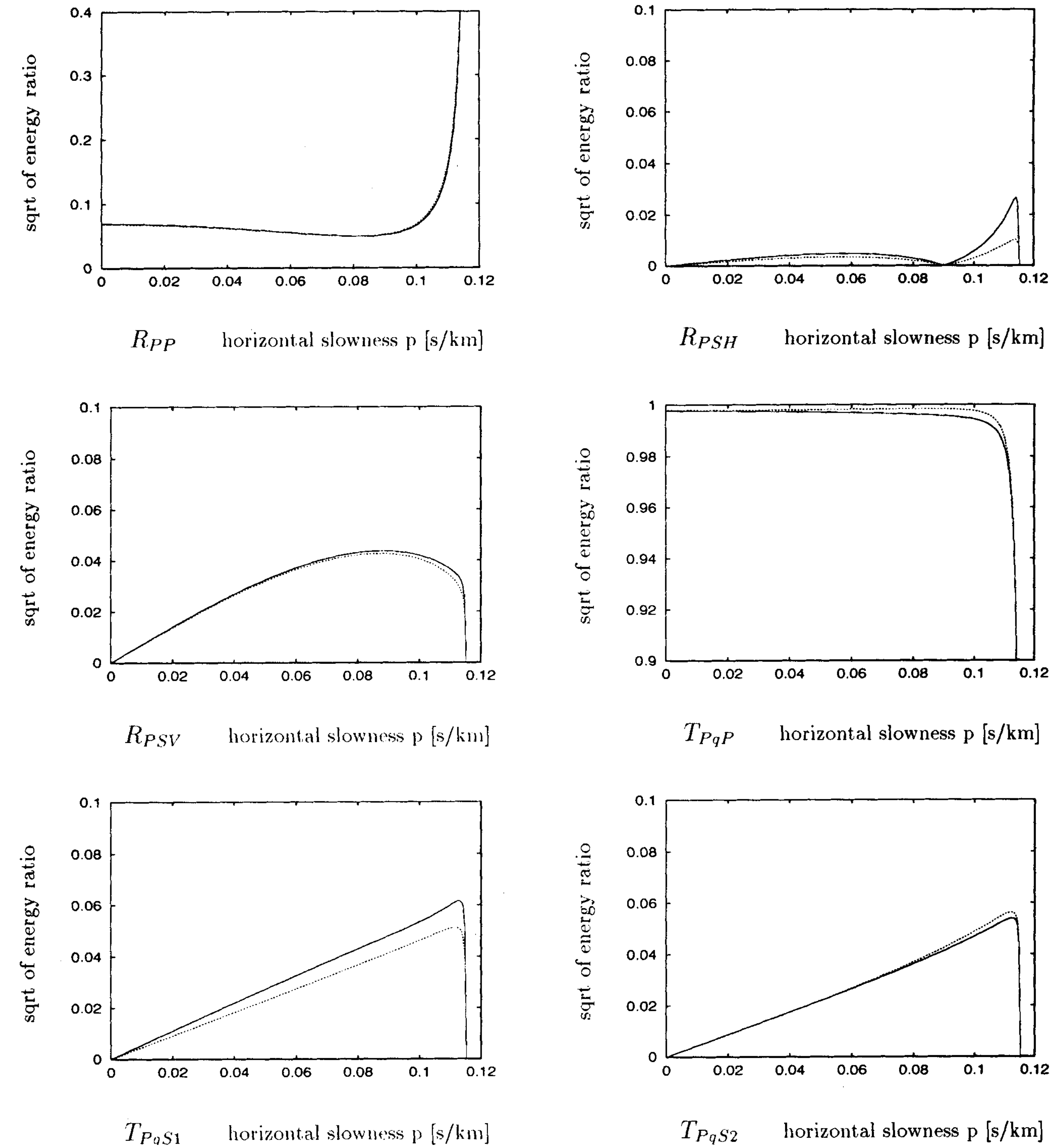

Figure 1. Reflection at an isotropic–hexagonal interface.

to first order given by eqs (39) and (43). In contrast to the original figures, the parametrization is with the horizontal slowness and not with the incidence angle.

There is a good general agreement between the exact and approximate results. The deviations are introduced by the approximate solution (30) of the eigenvalue problem. Note that the anisotropy of the hexagonal half-space is not very weak.

CONCLUSIONS

We have derived explicit expressions for reflection and transmission coefficients, which are an approximation for reflection between two general anisotropic half-spaces. The expressions contain not only a linear first-order correction to the isotropic coefficients but also a linear combination of isotropic shear-wave coefficients in zero order. The approximation is invalid near critical angles, near shear-wave singularities and when the anisotropy is too strong.

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APPENDIX A: THE ELEMENTS OF THE MATRICES IN EQS (13), (16), (18), (35) AND (43)

The solution of the isotropic reference eigenvalue problem is

$$\mathbf{A}^{(0)} = \begin{pmatrix} 0 & 0 & -p & -\frac{1}{\mu} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{\mu} & 0 \\ -p\frac{\lambda}{\lambda+2\mu} & 0 & 0 & 0 & 0 & -\frac{1}{\lambda+2\mu} \\ 4p^2\left(\mu - \frac{\mu^2}{\lambda+2\mu}\right) - \rho & 0 & 0 & 0 & 0 & -p\frac{\lambda}{\lambda+2\mu} \\ 0 & p^2\mu-\rho & 0 & 0 & 0 & 0 \\ 0 & 0 & -\rho & -p & 0 & 0 \end{pmatrix}, \tag{A1}$$

$$\mathbf{Q}^{(0)} = \mathrm{diag}(q_P, q_S, q_S, -q_P, -q_S, -q_S), \tag{A2}$$

$$\mathbf{Y}^{(0)} = \begin{pmatrix} p & 0 & -q_S & p & 0 & q_S \\ 0 & 1 & 0 & 0 & 1 & 0 \\ q_P & 0 & p & -q_P & 0 & p \\ -2\mu p q_P & 0 & \rho - 2\mu p^2 & 2\mu p q_P & 0 & \rho - 2\mu p^2 \\ 0 & -\mu q_S & 0 & 0 & \mu q_S & 0 \\ 2\mu p^2 - \rho & 0 & -2\mu p q_S & 2\mu p^2 - \rho & 0 & 2\mu p q_S \end{pmatrix}, \tag{A3}$$

$$\mathbf{Y}^{(0)-1} = \begin{pmatrix} \frac{p\mu}{\rho} & 0 & \frac{\rho - 2p^2\mu}{2q_P\rho} & -\frac{p}{2q_P\rho} & 0 & -\frac{1}{2\rho} \\ 0 & \frac{1}{2} & 0 & 0 & -\frac{1}{2\mu q_S} & 0 \\ \frac{2p^2\mu - \rho}{2q_S\rho} & 0 & \frac{p\mu}{\rho} & \frac{1}{2\rho} & 0 & -\frac{p}{2q_S\rho} \\ \frac{p\mu}{\rho} & 0 & \frac{2p^2\mu - \rho}{2q_P\rho} & \frac{p}{2q_P\rho} & 0 & -\frac{1}{2\rho} \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2\mu q_S} & 0 \\ \frac{\rho - 2p^2\mu}{2q_S\rho} & 0 & \frac{p\mu}{\rho} & \frac{1}{2\rho} & 0 & \frac{p}{2q_S\rho} \end{pmatrix}, \tag{A4}$$

$$\mathbf{X}^{(0)} = \mathbf{Y}^{(0)}\mathbf{L}, \qquad \mathbf{X}^{(0)-1} = \mathbf{L}^{-1}\mathbf{Y}^{(0)-1}, \tag{A5}$$

$$\mathbf{L} = \mathrm{diag}\left(\frac{1}{i\sqrt{2\rho q_P}}, \frac{1}{i\sqrt{2\mu q_S}}, \frac{1}{i\sqrt{2\rho q_S}}, \frac{1}{\sqrt{2\rho q_P}}, \frac{1}{\sqrt{2\mu q_S}}, \frac{1}{\sqrt{2\rho q_S}}\right). \tag{A6}$$

$$q_P = \sqrt{\frac{\rho}{\lambda+2\mu} - p^2}, \quad \sqrt{q_P} > 0\colon\; p^2 < \frac{\rho}{\lambda+2\mu}; \qquad q_P = i\sqrt{p^2 - \frac{\rho}{\lambda+2\mu}}, \quad \mathscr{Im}\sqrt{q_P} > 0\colon \quad p^2 > \frac{\rho}{\lambda+2\mu};$$

$$q_S = \sqrt{\frac{\rho}{\mu} - p^2}, \quad \sqrt{q_S} > 0\colon\; p^2 < \frac{\rho}{\mu}; \qquad q_S = i\sqrt{p^2 - \frac{\rho}{\mu}}, \quad \mathscr{Im}\sqrt{q_S} > 0\colon\; p^2 > \frac{\rho}{\mu}.$$

The perturbation part $\mathbf{A}^{(1)}$ of the system matrix is

$$A_{11}^{(1)} = -\frac{p}{\mu}c_{15}^{(1)} + \frac{p\lambda}{\mu(\lambda+2\mu)}c_{35}^{(1)}, \qquad A_{21}^{(1)} = -\frac{p}{\mu}c_{14}^{(1)} + \frac{p\lambda}{\mu(\lambda+2\mu)}c_{34}^{(1)}, \qquad A_{31}^{(1)} = -\frac{p}{\lambda+2\mu}c_{13}^{(1)} + \frac{p\lambda}{(\lambda+2\mu)^2}c_{33}^{(1)}, \tag{A7}$$

$$A_{12}^{(1)} = -\frac{p}{\mu}c_{56}^{(1)}, \qquad A_{22}^{(1)} = -\frac{p}{\mu}c_{46}^{(1)}, \qquad A_{32}^{(1)} = -\frac{p}{\lambda+2\mu}c_{36}^{(1)},$$

$$A_{13}^{(1)} = 0, \qquad A_{23}^{(1)} = 0, \qquad A_{33}^{(1)} = 0,$$

$$A_{14}^{(1)} = \frac{1}{\mu^2}c_{55}^{(1)}, \qquad A_{15}^{(1)} = \frac{1}{\mu^2}c_{45}^{(1)}, \qquad A_{25}^{(1)} = \frac{1}{\mu^2}c_{44}^{(1)},$$

$$A_{16}^{(1)} = \frac{1}{\mu(\lambda+2\mu)}c_{35}^{(1)}, \qquad A_{26}^{(1)} = \frac{1}{\mu(\lambda+2\mu)}c_{34}^{(1)}, \qquad A_{36}^{(1)} = \frac{1}{(\lambda+2\mu)^2}c_{33}^{(1)},$$

$$A_{41}^{(1)} = p^2\left(c_{11}^{(1)} - \frac{2\lambda}{\lambda+2\mu}c_{13}^{(1)} + \frac{\lambda^2}{(\lambda+2\mu)^2}c_{33}^{(1)}\right), \qquad A_{42}^{(1)} = p^2\left(c_{16}^{(1)} - \frac{\lambda}{\lambda+2\mu}c_{36}^{(1)}\right), \qquad A_{52}^{(1)} = p^2 c_{66}^{(1)},$$

$$A_{43}^{(1)} = 0, \qquad A_{53}^{(1)} = 0, \qquad A_{63}^{(1)} = 0.$$

The elements of the perturbation matrix $\mathbf{S}$ are

$$S_{11} = -\frac{1}{2q_P\rho}[p^4c_{11}^{(1)} + 2p^2q_P^2c_{13}^{(1)} + 4p^3q_Pc_{15}^{(1)} + q_P^4c_{33}^{(1)} + 4pq_P^3c_{35}^{(1)} + 4p^2q_P^2c_{55}^{(1)}], \tag{A8}$$

$$S_{12} = -\frac{1}{2q_P\rho}\sqrt{\frac{\rho q_P}{\mu q_S}}[p^2q_Sc_{14}^{(1)} + p^3c_{16}^{(1)} + q_P^2q_Sc_{34}^{(1)} + pq_P^2c_{36}^{(1)} + 2pq_Pq_Sc_{45}^{(1)} + 2p^2q_Pc_{56}^{(1)}],$$

$$S_{13} = -\frac{1}{2q_P\rho}\sqrt{\frac{q_P}{q_S}}[-p^3q_Sc_{11}^{(1)} - pq_S(q_P^2-p^2)c_{13}^{(1)} - p^2(q_S^2-p^2+2q_Pq_S)c_{15}^{(1)} + pq_P^2q_Sc_{33}^{(1)}$$
$$+ (2p^2q_Pq_S - q_P^2(q_S^2-p^2))c_{35}^{(1)} - 2pq_P(q_S^2-p^2)c_{55}^{(1)}],$$

$$S_{14} = -\frac{1}{2q_P\rho}i[p^4c_{11}^{(1)} + 2p^2q_P^2c_{13}^{(1)} + q_P^4c_{33}^{(1)} - 4p^2q_P^2c_{55}^{(1)}],$$

$$S_{15} = -\frac{1}{2q_P\rho}i\sqrt{\frac{\rho q_P}{\mu q_S}}[-p^2q_Sc_{14}^{(1)} + p^3c_{16}^{(1)} - q_P^2q_Sc_{34}^{(1)} + pq_P^2c_{36}^{(1)} - 2pq_Pq_Sc_{45}^{(1)} + 2p^2q_Pc_{56}^{(1)}],$$

$$S_{16} = -\frac{1}{2q_P\rho}i\sqrt{\frac{q_P}{q_S}}[p^3q_Sc_{11}^{(1)} + pq_S(q_P^2-p^2)c_{13}^{(1)} - p^2(q_S^2-p^2-2q_Pq_S)c_{15}^{(1)}$$
$$- pq_P^2q_Sc_{33}^{(1)} - (2p^2q_Pq_S + q_P^2(q_S^2-p^2))c_{35}^{(1)} - 2pq_P(q_S^2-p^2)c_{55}^{(1)}],$$

$$S_{22} = -\frac{1}{2q_S\mu}[q_S^2c_{44}^{(1)} + 2pq_Sc_{46}^{(1)} + p^2c_{66}^{(1)}],$$

$$S_{23} = -\frac{1}{2q_S\mu}\sqrt{\frac{\mu}{\rho}}[-pq_S^2c_{14}^{(1)} - p^2q_Sc_{16}^{(1)} + pq_S^2c_{34}^{(1)} + p^2q_Sc_{36}^{(1)} - q_S(q_S^2-p^2)c_{45}^{(1)} - p(q_S^2-p^2)c_{56}^{(1)}],$$

$$S_{24} = -\frac{1}{2q_S\mu}i\sqrt{\frac{\mu q_S}{\rho q_P}}[p^2q_Sc_{14}^{(1)} + p^3c_{16}^{(1)} + q_P^2q_Sc_{34}^{(1)} + pq_P^2c_{36}^{(1)} - 2pq_Pq_Sc_{45}^{(1)} - 2p^2q_Pc_{56}^{(1)}],$$

$$S_{25} = -\frac{1}{2q_S\mu}i[-q_S^2c_{44}^{(1)} + p^2c_{66}^{(1)}],$$

$$S_{26} = -\frac{1}{2q_S\mu}i\sqrt{\frac{\mu}{\rho}}[pq_S^2c_{14}^{(1)} + p^2q_Sc_{16}^{(1)} - pq_S^2c_{34}^{(1)} - p^2q_Sc_{36}^{(1)} - q_S(q_S^2-p^2)c_{45}^{(1)} - p(q_S^2-p^2)c_{56}^{(1)}],$$

$$S_{33} = -\frac{1}{2q_S\rho}[p^2q_S^2c_{11}^{(1)} - 2p^2q_S^2c_{13}^{(1)} + 2pq_S(q_S^2 - p^2)c_{15}^{(1)} + p^2q_S^2c_{33}^{(1)} - 2pq_S(q_S^2 - p^2)c_{35}^{(1)} + (q_S^2 - p^2)^2c_{55}^{(1)}],$$

$$S_{34} = -\frac{1}{2q_S\rho}i\sqrt{\frac{q_S}{q_P}}[-p^3q_Sc_{11}^{(1)} - pq_S(q_P^2 - p^2)c_{13}^{(1)} - p^2(q_S^2 - p^2 - 2q_Pq_S)c_{15}^{(1)} + pq_P^2q_Sc_{33}^{(1)}$$
$$-(q_P^2(q_S^2 - p^2) + 2p^2q_Pq_S)c_{35}^{(1)} + 2pq_P(q_S^2 - p^2)c_{55}^{(1)}],$$

$$S_{35} = -\frac{1}{2q_S\rho}i\sqrt{\frac{\rho}{\mu}}[pq_S^2c_{14}^{(1)} - p^2q_Sc_{16}^{(1)} - pq_S^2c_{34}^{(1)} + p^2q_Sc_{36}^{(1)} + q_S(q_S^2 - p^2)c_{45}^{(1)} - p(q_S^2 - p^2)c_{56}^{(1)}],$$

$$S_{36} = -\frac{1}{2q_S\rho}i[-p^2q_S^2c_{11}^{(1)} + 2p^2q_S^2c_{13}^{(1)} - p^2q_S^2c_{33}^{(1)} + (q_S^2 - p^2)^2c_{55}^{(1)}],$$

$$S_{44} = -\frac{1}{2q_P\rho}[-p^4c_{11}^{(1)} - 2p^2q_P^2c_{13}^{(1)} + 4p^3q_Pc_{15}^{(1)} - q_P^4c_{33}^{(1)} + 4pq_P^3c_{35}^{(1)} - 4p^2q_P^2c_{55}^{(1)}],$$

$$S_{45} = -\frac{1}{2q_P\rho}\sqrt{\frac{\rho q_P}{\mu q_S}}[p^2q_Sc_{14}^{(1)} - p^3c_{16}^{(1)} + q_P^2q_Sc_{34}^{(1)} - pq_P^2c_{36}^{(1)} - 2pq_Pq_Sc_{45}^{(1)} + 2p^2q_Pc_{56}^{(1)}],$$

$$S_{46} = -\frac{1}{2q_P\rho}\sqrt{\frac{q_P}{q_S}}[-p^3q_Sc_{11}^{(1)} - pq_S(q_P^2 - p^2)c_{13}^{(1)} + p^2(q_S^2 - p^2 + 2q_Pq_S)c_{15}^{(1)} + pq_P^2q_Sc_{33}^{(1)}$$
$$+(q_P^2(q_S^2 - p^2) - 2p^2q_Pq_S)c_{35}^{(1)} - 2pq_P(q_S^2 - p^2)c_{55}^{(1)}],$$

$$S_{55} = -\frac{1}{2q_S\mu}[-q_S^2c_{44}^{(1)} + 2pq_Sc_{46}^{(1)} - p^2c_{66}^{(1)}],$$

$$S_{56} = -\frac{1}{2q_S\mu}\sqrt{\frac{\mu}{\rho}}[pq_S^2c_{14}^{(1)} - p^2q_Sc_{16}^{(1)} - pq_S^2c_{34}^{(1)} + p^2q_Sc_{36}^{(1)} - q_S(q_S^2 - p^2)c_{45}^{(1)} + p(q_S^2 - p^2)c_{56}^{(1)}],$$

$$S_{66} = -\frac{1}{2q_S\rho}[-p^2q_S^2c_{11}^{(1)} + 2p^2q_S^2c_{13}^{(1)} + 2pq_S(q_S^2 - p^2)c_{15}^{(1)} - p^2q_S^2c_{33}^{(1)} - 2pq_S(q_S^2 - p^2)c_{35}^{(1)} - (q_S^2 - p^2)^2c_{55}^{(1)}].$$

The matrix $\mathbf{N}^{(0)-1}$ has elements

$$N_{11}^{(0)-1} = \frac{1}{\Delta_{PSV}}i\sqrt{2\rho_1q_{P1}}\{p^5[-4\mu_2(\mu_2 - \mu_1)] + p^3[-2\mu_1\rho_2 - 2\mu_2\rho_1 + 4\mu_2\rho_2 - 4\mu_2(\mu_2 - \mu_1)q_{P2}q_{S2}]$$
$$+ p[-2\mu_2\rho_1q_{P2}q_{S2} - 2\mu_1\rho_2q_{P2}q_{S1} - \rho_2(\rho_2 - \rho_1)]\}, \quad \text{(A9)}$$

$$N_{21}^{(0)-1} = 0,$$

$$N_{31}^{(0)-1} = \frac{1}{\Delta_{PSV}}i\sqrt{2\rho_1q_{S1}}\{p^4[4\mu_2(\mu_2 - \mu_1)q_{P1}] + p^2[-4\mu_2\rho_2q_{P1} - 2\mu_1\rho_2(q_{P2} - q_{P1}) + 4\mu_2(\mu_2 - \mu_1)q_{P1}q_{P2}q_{S2}] + \rho_1\rho_2q_{P2} + \rho_2^2q_{P1}\},$$

$$N_{41}^{(0)-1} = \frac{1}{\Delta_{PSV}}\sqrt{2\rho_2q_{P2}}\{p^5[-4\mu_1(\mu_2 - \mu_1)] + p^3[-4\mu_1\rho_1 + 2\mu_2\rho_1 + 2\mu_1\rho_2 - 4\mu_1(\mu_2 - \mu_1)q_{P1}q_{S1}]$$
$$+ p[-\rho_1(\rho_2 - \rho_1) + 2\mu_2\rho_1q_{P1}q_{S2} + 2\mu_1\rho_2q_{P1}q_{S1}]\},$$

$$N_{51}^{(0)-1} = 0,$$

$$N_{61}^{(0)-1} = \frac{1}{\Delta_{PSV}}\sqrt{2\rho_2q_{S2}}\{p^4[-4\mu_1(\mu_2 - \mu_1)q_{P2}] + p^2[-4\mu_1\rho_1q_{P2} + 2\mu_2\rho_1(q_{P2} - q_{P1}) - 4\mu_1(\mu_2 - \mu_1)q_{P1}q_{P2}q_{S1}] + \rho_1^2q_{P2} + \rho_1\rho_2q_{P1}\},$$

$$N_{12}^{(0)-1} = 0,$$

$$N_{22}^{(0)-1} = \frac{1}{\Delta_{SH}}i\sqrt{2\mu_1q_{S1}}\{\mu_2q_{S2}\},$$

$$N_{32}^{(0)-1} = 0,$$

$$N_{42}^{(0)-1} = 0,$$

$$N_{52}^{(0)-1} = \frac{1}{\Delta_{SH}}\sqrt{2\mu_2q_{S2}}\{-\mu_1q_{S1}\},$$

$$N_{62}^{(0)-1} = 0,$$

$$N_{13}^{(0)-1} = \frac{1}{\Delta_{PSV}} i\sqrt{2\rho_1 q_{P1}}\{p^4[-4\mu_2(\mu_2-\mu_1)q_{S1}] + p^2[-4(\mu_2-\mu_1)\mu_2 q_{P2}q_{S1}q_{S2} + 4\rho_2\mu_2 q_{S1} - 2\rho_2\mu_1 q_{S1} + 2\mu_1\rho_2 q_{S2}] - \rho_2^2 q_{S1} - \rho_1\rho_2 q_{S2}\},$$

$$N_{23}^{(0)-1} = 0,$$

$$N_{33}^{(0)-1} = \frac{1}{\Delta_{PSV}} i\sqrt{2\rho_1 q_{S1}}\{p^5[-4\mu_2(\mu_2-\mu_1)] + p^3[4\mu_2\rho_2 - 2\mu_1\rho_2 - 2\mu_2\rho_1 - 4\mu_2(\mu_2-\mu_1)q_{P2}q_{S2}] + p[-2\mu_2\rho_1 q_{P2}q_{S2} - 2\mu_1\rho_2 q_{P1}q_{S2} - \rho_2(\rho_2-\rho_1)]\},$$

$$N_{43}^{(0)-1} = \frac{1}{\Delta_{PSV}} \sqrt{2\rho_2 q_{P2}}\{p^4[4\mu_1(\mu_2-\mu_1)q_{S2}] + p^2[4\mu_1\rho_1 q_{S2} + 2\mu_2\rho_1 q_{S1} - 2\mu_2\rho_1 q_{S2} + 4\mu_1(\mu_2-\mu_1)q_{P1}q_{S1}q_{S2}] - \rho_1\rho_2 q_{S1} - \rho_1^2 q_{S2}\},$$

$$N_{53}^{(0)-1} = 0,$$

$$N_{63}^{(0)-1} = \frac{1}{\Delta_{PSV}} \sqrt{2\rho_2 q_{S2}}\{p^5[-4\mu_1(\mu_2-\mu_1)] + p^3[-4\mu_1\rho_1 + 2\mu_2\rho_1 + 2\mu_1\rho_2 - 4\mu_1(\mu_2-\mu_1)q_{P1}q_{S1}] + p[-\rho_1(\rho_2-\rho_1) + 2\mu_1\rho_2 q_{P1}q_{S1} + 2\mu_2\rho_1 q_{P2}q_{S1}]\},$$

$$N_{14}^{(0)-1} = \frac{1}{\Delta_{PSV}} i\sqrt{2\rho_1 q_{P1}}\{p^3[-2(\mu_2-\mu_1)q_{S1}] + p[\rho_2(q_{S1}+q_{S2}) - 2(\mu_2-\mu_1)q_{P2}q_{S1}q_{S2}]\},$$

$$N_{24}^{(0)-1} = 0,$$

$$N_{34}^{(0)-1} = \frac{1}{\Delta_{PSV}} i\sqrt{2\rho_1 q_{S1}}\{p^4[-2(\mu_2-\mu_1)] + p^2[\rho_2 - \rho_1 - 2(\mu_2-\mu_1)q_{P2}q_{S2}] - \rho_1 q_{P2}q_{S2} - \rho_2 q_{P1}q_{S2}\},$$

$$N_{44}^{(0)-1} = \frac{1}{\Delta_{PSV}} \sqrt{2\rho_2 q_{P2}}\{p^3[2(\mu_2-\mu_1)q_{S2}] + p[\rho_1(q_{S1}+q_{S2}) + 2(\mu_2-\mu_1)q_{P1}q_{S1}q_{S2}]\},$$

$$N_{54}^{(0)-1} = 0,$$

$$N_{64}^{(0)-1} = \frac{1}{\Delta_{PSV}} \sqrt{2\rho_2 q_{S2}}\{p^4[-2(\mu_2-\mu_1)] + p^2[\rho_2 - \rho_1 - 2(\mu_2-\mu_1)q_{P1}q_{S1}] + \rho_1 q_{P2}q_{S1} + \rho_2 q_{P1}q_{S1}\},$$

$$N_{15}^{(0)-1} = 0,$$

$$N_{25}^{(0)-1} = -\frac{1}{\Delta_{SH}} i\sqrt{2\mu_1 q_{S1}},$$

$$N_{35}^{(0)-1} = 0,$$

$$N_{45}^{(0)-1} = 0,$$

$$N_{55}^{(0)-1} = -\frac{1}{\Delta_{SH}} \sqrt{2\mu_2 q_{S2}},$$

$$N_{65}^{(0)-1} = 0,$$

$$N_{16}^{(0)-1} = \frac{1}{\Delta_{PSV}} i\sqrt{2\rho_1 q_{P1}}\{p^4[2(\mu_2-\mu_1)] + p^2[-(\rho_2-\rho_1) + 2(\mu_2-\mu_1)q_{P2}q_{S2}] + \rho_1 q_{P2}q_{S2} + \rho_2 q_{P2}q_{S1}\},$$

$$N_{26}^{(0)-1} = 0,$$

$$N_{36}^{(0)-1} = \frac{1}{\Delta_{PSV}} i\sqrt{2\rho_1 q_{S1}}\{p^3[-2(\mu_2-\mu_1)q_{P1}] + p[\rho_2(q_{P1}+q_{P2}) - 2(\mu_2-\mu_1)q_{P1}q_{P2}q_{S2}]\},$$

$$N_{46}^{(0)-1} = \frac{1}{\Delta_{PSV}} \sqrt{2\rho_2 q_{P2}}\{p^4[2(\mu_2-\mu_1)] + p^2[2(\mu_2-\mu_1)q_{P1}q_{S1} - (\rho_2-\rho_1)] - \rho_1 q_{P1}q_{S2} - \rho_2 q_{P1}q_{S1}\},$$

$$N_{56}^{(0)-1} = 0,$$

$$N_{66}^{(0)-1} = \frac{1}{\Delta_{PSV}} \sqrt{2\rho_2 q_{S2}}\{p^3[2(\mu_2-\mu_1)q_{P2}] + p[\rho_1(q_{P1}+q_{P2}) + 2(\mu_2-\mu_1)q_{P1}q_{P2}q_{S1}]\}.$$

$$\Delta_{PSV} = p^6[4(\mu_2-\mu_1)^2] + p^4[4(\mu_2-\mu_1)(-(\rho_2-\rho_1) + (\mu_2-\mu_1)(q_{P1}q_{S1}+q_{P2}q_{S2}))] \tag{A10}$$

$$+ p^2[4(\mu_2-\mu_1)(\rho_1 q_{P2}q_{S2} - \rho_2 q_{P1}q_{S1} + (\mu_2-\mu_1)q_{P1}q_{P2}q_{S1}q_{S2}) + (\rho_2-\rho_1)^2]$$

$$+ \rho_1^2 q_{P2}q_{S2} + \rho_2^2 q_{P1}q_{S1} + \rho_1\rho_2(q_{P1}q_{S2}+q_{P2}q_{S1}),$$

$$\Delta_{SH} = -\mu_2 q_{S2} - \mu_1 q_{S1}.$$

The isotropic reflection coefficients are

$$R_{PP} = \frac{1}{\Delta_{PSV}} i\{p^6[-4(\mu_2-\mu_1)^2] + p^4[4(\mu_2-\mu_1)^2_{P1}q_{S1} - q_{P2}q_{S2}) + 4(\mu_2-\mu_1)(\rho_2-\rho_1)] \tag{A11}$$

$$+ p^2[2(\mu_2-\mu_1)(-2\rho_2 q_{P1}q_{S1} - 2\rho_1 q_{P2}q_{S2}) - (\rho_2-\rho_1)^2 + 4(\mu_2-\mu_1)^2 q_{P1}q_{P2}q_{S1}q_{S2}]$$

$$+ \rho_2^2 q_{P1}q_{S1} - \rho_1^2 q_{P2}q_{S2} + \rho_1\rho_2(q_{P1}q_{S2} - q_{P2}q_{S1})\},$$

$$R_{PSH} = 0,$$

$$R_{PSV} = \frac{1}{\Delta_{PSV}} i\sqrt{\frac{q_{S1}}{q_{P1}}}\{p^5[8(\mu_2-\mu_1)^2 q_{P1}] + p^3[-8(\mu_2-\mu_1)\rho_2 q_{P1} + 8(\mu_2-\mu_1)^2 q_{P1}q_{P2}q_{S2} + 4(\mu_2-\mu_1)\rho_1 q_{P1}]$$

$$+ p[2\rho_2(\rho_2-\rho_1)q_{P1} + 4(\mu_2-\mu_1)\rho_1 q_{P1}q_{P2}q_{S2}]\},$$

$$T_{PP} = \frac{1}{\Delta_{PSV}}\sqrt{\frac{q_{P2}\rho_2}{q_{P1}\rho_1}}\{p^2[4(\mu_2-\mu_1)\rho_1 q_{P1}(q_{S2}-q_{S1})] + 2\rho_1^2 q_{P1}q_{S2} + 2\rho_1\rho_2 q_{P1}q_{S1}\},$$

$$T_{PSH} = 0,$$

$$T_{PSV} = \frac{1}{\Delta_{PSV}}\sqrt{\frac{\rho_2 q_{S2}}{\rho_1 q_{P1}}}\{p^3[-4(\mu_2-\mu_1)\rho_1 q_{P1}] + p[2\rho_1(\rho_2-\rho_1)q_{P1} - 4(\mu_2-\mu_1)\rho_1 q_{P1}q_{P2}q_{S1}]\},$$

$$R_{SVP} = \frac{1}{\Delta_{PSV}} i\sqrt{\frac{q_{P1}}{q_{S1}}}\{p^5[-8(\mu_2-\mu_1)^2 q_{S1}] + p^3[8\rho_2(\mu_2-\mu_1)q_{S1} - 4(\mu_2-\mu_1)\rho_1 q_{S1} - 8(\mu_2-\mu_1)^2 q_{P2}q_{S1}q_{S2}]$$

$$+ p[-4(\mu_2-\mu_1)\rho_1 q_{P2}q_{S1}q_{S2} - 2\rho_2(\rho_2-\rho_1)q_{S1}]\},$$

$$R_{SVSH} = 0,$$

$$R_{SVSV} = \frac{1}{\Delta_{PSV}} i\{p^6[-4(\mu_2-\mu_1)^2] + p^4[-4(\mu_2-\mu_1)^2(q_{P2}q_{S2} - q_{P1}q_{S1}) + 4(\mu_2-\mu_1)(\rho_2-\rho_1)]$$

$$+ p^2[-4(\mu_2-\mu_1)(\rho_1 q_{P2}q_{S2} + \rho_2 q_{P1}q_{S1}) + 4(\mu_2-\mu_1)^2 q_{P1}q_{P2}q_{S1}q_{S2} + (\rho_2-\rho_1)^2]$$

$$+ \rho_2^2 q_{P1}q_{S1} - \rho_1^2 q_{P2}q_{S2} + \rho_1\rho_2(q_{P2}q_{S1} - q_{P1}q_{S2})\},$$

$$T_{SVP} = \frac{1}{\Delta_{PSV}}\sqrt{\frac{\rho_2 q_{P2}}{\rho_1 q_{S1}}}\{p^3[4(\mu_2-\mu_1)\rho_1 q_{S1}] + p[4(\mu_2-\mu_1)\rho_1 q_{P1}q_{S1}q_{S2} - 2\rho_1(\rho_2-\rho_1)q_{S1}]\},$$

$$T_{SVSH} = 0,$$

$$T_{SVSV} = \frac{1}{\Delta_{PSV}}\sqrt{\frac{\rho_2 q_{S2}}{\rho_1 q_{S1}}}\{p^2[4(\mu_2-\mu_1)\rho_1 q_{S1}(q_{P2}-q_{P1})] + 2\rho_1 q_{S1}(\rho_2 q_{P1} + \rho_1 q_{P2})\},$$

$$R_{SHP} = 0,$$

$$R_{SHSH} = \frac{1}{\Delta_{SH}} i\{\mu_2 q_{S2} - \mu_1 q_{S1}\},$$

$$R_{SHSV} = 0.$$

$$T_{SHP} = 0,$$

$$T_{SHSH} = \frac{1}{\Delta_{SH}}\sqrt{\frac{\mu_2 q_{S2}}{\mu_1 q_{S1}}}\{-2\mu_1 q_{S1}\},$$

$$T_{SHSV} = 0.$$